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LETTER TO THE EDITOR

Analytic Bethe ansatz and T-system in $C_2^{(1)}$ vertex models

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Abstract. Eigenvalues of the commuting family of transfer matrices are expected to obey the T-system, a set of functional relation, proposed recently. Here we obtain the solution to the T-system for $C_2^{(1)}$ vertex models. They are compatible with the analytic Bethe ansatz and Yang–Baxterize the classical characters.

Solvable lattice models in two-dimensions possess a commuting family of the row-to-row transfer matrices [1]. Recently, a set of functional relations (FRs), the T-system are proposed among them [2] for a wide class of models associated with any classical simple Lie algebra or its quantum affine analogue [3,4]. In the QISM terminology [5], *the T-system relates the transfer matrices with various fusion types in the auxiliary space* but acting on a common quantum space. It generalizes earlier FRs [6–9] and enables the calculation of various physical quantities [10]. The structure that underlies the T-system is an (short) exact sequence of the finite dimensional modules of the above mentioned algebras [2]. As discussed therein, there is an intriguing connection between the T-system, the thermodynamic Bethe ansatz (TBA) and dilogarithm identities, indicating some deep interplay among these subjects.

In this letter we report the solution to the $C_2^{(1)}$ T-system that is compatible with the analytic Bethe ansatz [11, 12] and Yang–Baxterizes the classical characters. To explain the problem, let $W_m^{(a)}$ ($a = 1, 2, m \in \mathbf{Z}_{\geqslant 1}$) be the irreducible finite dimensional representation (IFDR) of the quantum affine algebra $U_q(C_2^{(1)})$ (q: generic) as sketched in section 3.2 of [2]. As the C_2-module, it decomposes as

$$\begin{aligned} W_m^{(1)} &\simeq V_{m\omega_1} \oplus V_{(m-2)\omega_1} \oplus \dots \oplus \begin{cases} V_0 & m \text{ even} \\ V_{\omega_1} & m \text{ odd} \end{cases} \\ W_m^{(2)} &\simeq V_{m\omega_2} \end{aligned} \tag{1}$$

where ω_1, ω_2 are the fundamental weights and V_ω denotes the IFDR of C_2 with highest weight ω. Thus $\dim W_m^{(1)} = (m+2)(m+4)(m^2+6m+6)/48$ for m even, $= (m+1)(m+3)^2(m+5)/48$ for m odd and $\dim W_m^{(2)} = (m+1)(m+2)(2m+3)/6$. For $W, W' \in \{W_m^{(a)} |,\ a = 1, 2, m \in \mathbf{Z}_{\geqslant 1}\}$, there exists the quantum R-matrix $R_{W,W'}(u)$ acting on $W \otimes W'$ and satisfying the Yang–Baxter equation

$$R_{W,W'}(u)R_{W,W''}(u+v)R_{W',W''}(v) = R_{W',W''}(v)R_{W,W''}(u+v)R_{W,W'}(u) \tag{2}$$

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with $u, v \in \mathbf{C}$ being the spectral parameters. For $W = W' = W_1^{(1)}$, the R-matrix has been explicitly written down in [13, 14], from which all the other $R_{W,W'}$ may be constructed via the fusion procedure [15]. $W_m^{(a)}$ is an analogue of the m-fold symmetric tensor representation of $W_1^{(a)}$. The transfer matrix with auxiliary space $W_m^{(a)}$ is then defined by

$$T_m^{(a)}(u) = \mathrm{Tr}_{W_m^{(a)}}(R_{W_m^{(a)},W_s^{(p)}}(u - w_1)\dots R_{W_m^{(a)},W_s^{(p)}}(u - w_N)) \tag{3}$$

up to an overall scalar multiple. Here $N \in 2\mathbf{Z}$ denotes the system size, $w_1, \dots w_N$ are complex parameters representing the inhomogeneity, $p = 1, 2$ and $s \in \mathbf{Z}_{\geqslant 1}$. We say that (3) is the row-to-row transfer matrix with fusion type $W_m^{(a)}$ acting on the quantum space $(W_s^{(p)})^{\otimes N}$. We shall reserve the letters p and s for this meaning throughout. Thanks to the Yang–Baxter equation (2), the transfer matrices (3) form a commuting family

$$[T_m^{(a)}(u), T_{m'}^{(a')}(u')] = 0. \tag{4}$$

We shall write the eigenvalues of $T_m^{(a)}(u)$ as $\Lambda_m^{(a)}(u)$. Our goal is to find an explicit formula for them.

For the purpose, we postulate the (unrestricted) T-system [2]:

$$T_{2m}^{(1)}\left(u-\frac{1}{2}\right)T_{2m}^{(1)}\left(u+\frac{1}{2}\right) = T_{2m+1}^{(1)}(u)T_{2m-1}^{(1)}(u) + g_{2m}^{(1)}(u)T_m^{(2)}\left(u-\frac{1}{2}\right)T_m^{(2)}\left(u+\frac{1}{2}\right) \tag{5a}$$

$$T_{2m+1}^{(1)}\left(u-\frac{1}{2}\right)T_{2m+1}^{(1)}\left(u+\frac{1}{2}\right) = T_{2m+2}^{(1)}(u)T_{2m}^{(1)}(u) + g_{2m+1}^{(1)}(u)T_m^{(2)}(u)T_{m+1}^{(2)}(u) \tag{5b}$$

$$T_m^{(2)}(u-1)T_m^{(2)}(u+1) = T_{m+1}^{(2)}(u)T_{m-1}^{(2)}(u) + g_m^{(2)}(u)T_{2m}^{(1)}(u). \tag{5c}$$

Here $g_m^{(a)}(u)$ is a scalar function that depends on $W_s^{(p)}$ and overall normalization of the transfer matrices. Due to (4) the eigenvalues $\Lambda_m^{(a)}(u)$ also obey the same system as (5), which can be solved successively yielding an expression of the $\Lambda_m^{(a)}(u)$ in terms of $\Lambda_1^{(1)}(u + \text{shift})$ and $\Lambda_1^{(2)}(u + \text{shift})$. Thus the first step to achieve the goal is to find the formula for the eigenvalues $\Lambda_1^{(1)}(u)$ and $\Lambda_1^{(2)}(u)$. This we do by the analytic Bethe ansatz. The method consists of assuming the so-called 'dressed vacuum form' for the eigenvalues and determining the unknown parts thereby introduced from some functional properties and asymptotic behaviours. See [12, 16] for the detail. To present the results for our problem, we prepare a few notations. Let α_1, α_2 be the simple roots of C_2. We take α_2 to be a long root and normalize it as $(\alpha_2 \mid \alpha_2) = 2$ via the bilinear form $(|)$. Then one has $(\alpha_a \mid \omega_b) = \delta_{ab}/t_a$, where $t_1 = 2$, $t_2 = 1$. We set

$$\phi(u) = \prod_{j=1}^{N}[u - w_j] \qquad [u] = q^u - q^{-u}$$

$$\phi_m^{(a)}(u) = \phi\left(u+\frac{m-1}{t_a}\right)\phi\left(u+\frac{m-3}{t_a}\right)\dots\phi\left(u-\frac{m-1}{t_a}\right) \qquad a = 1, 2, m \in \mathbf{Z}_{\geqslant 1} \tag{6}$$

$$Q_a(u) = \prod_{j=1}^{N_a}[u - iu_j^{(a)}] \qquad a = 1, 2.$$

Here N_1, N_2 are non-negative integers such that $\omega^{(p)} \stackrel{\text{def}}{=} Ns\omega_p - N_1\alpha_1 - N_2\alpha_2$ is a non-negative weight. The numbers $\{u_j^{(a)} \mid a = 1, 2, 1 \leqslant j \leqslant N_a\}$ are the solutions to the Bethe ansatz equation [16]

$$-\frac{\phi(iu_k^{(a)} + (s/t_p)\delta_{pa})}{\phi(iu_k^{(a)} - (s/t_p)\delta_{pa})} = \prod_{b=1}^{2} \frac{Q_b(iu_k^{(a)} + (\alpha_a \mid \alpha_b))}{Q_b(iu_k^{(a)} - (\alpha_a \mid \alpha_b))} \qquad a = 1, 2, 1 \leqslant k \leqslant N_a. \tag{7}$$

Under these definitions, the result of the analytic Bethe ansatz reads as follows.

Case $p = 1$;

$$\begin{aligned}\Lambda_1^{(1)}(u) &= \phi_s^{(1)}(u+3)\phi_s^{(1)}(u+1)\frac{Q_1(u-\frac{1}{2})}{Q_1(u+\frac{1}{2})} + \phi_s^{(1)}(u+2)\phi_s^{(1)}(u)\frac{Q_1(u+\frac{7}{2})}{Q_1(u+\frac{5}{2})} \\ &\quad + \phi_s^{(1)}(u+3)\phi_s^{(1)}(u)\left(\frac{Q_1(u+\frac{3}{2})Q_2(u-\frac{1}{2})}{Q_1(u+\frac{1}{2})Q_2(u+\frac{3}{2})} + \frac{Q_1(u+\frac{3}{2})Q_2(u+\frac{7}{2})}{Q_1(u+\frac{5}{2})Q_2(u+\frac{3}{2})}\right)\end{aligned} \tag{8a}$$

$$\begin{aligned}\Lambda_1^{(2)}(u) &= \phi_s^{(1)}\left(u+\frac{5}{2}\right)\left(\frac{Q_2(u-1)}{Q_2(u+1)} + \frac{Q_1(u)Q_2(u+3)}{Q_1(u+2)Q_2(u+1)}\right) \\ &\quad + \phi_s^{(1)}\left(u+\frac{1}{2}\right)\left(\frac{Q_2(u+4)}{Q_2(u+2)} + \frac{Q_1(u+3)Q_2(u)}{Q_1(u+1)Q_2(u+2)}\right) \\ &\quad + \phi_s^{(1)}\left(u+\frac{3}{2}\right)\frac{Q_1(u)Q_1(u+3)}{Q_1(u+1)Q_1(u+2)}.\end{aligned} \tag{8b}$$

Case $p = 2$;

$$\begin{aligned}\Lambda_1^{(1)}(u) &= \phi_s^{(2)}\left(u+\frac{5}{2}\right)\phi_s^{(2)}\left(u+\frac{3}{2}\right)\left(\frac{Q_1(u-\frac{1}{2})}{Q_1(u+\frac{1}{2})} + \frac{Q_1(u+\frac{3}{2})Q_2(u-\frac{1}{2})}{Q_1(u+\frac{1}{2})Q_2(u+\frac{3}{2})}\right) \\ &\quad + \phi_s^{(2)}\left(u+\frac{3}{2}\right)\phi_s^{(2)}\left(u+\frac{1}{2}\right)\left(\frac{Q_1(u+\frac{3}{2})Q_2(u+\frac{7}{2})}{Q_1(u+\frac{5}{2})Q_2(u+\frac{3}{2})} + \frac{Q_1(u+\frac{7}{2})}{Q_1(u+\frac{5}{2})}\right)\end{aligned} \tag{8c}$$

$$\begin{aligned}\Lambda_1^{(2)}(u) &= \phi_2^{(2)}(u+3)\phi_s^{(2)}(u+2)\frac{Q_2(u-1)}{Q_2(u+1)} + \phi_s^{(2)}(u+1)\phi_s^{(2)}(u)\frac{Q_2(u+4)}{Q_2(u+2)} \\ &\quad + \phi_s^{(2)}(u+3)\phi_s^{(2)}(u)\left(\frac{Q_1(u)Q_2(u+3)}{Q_1(u+2)Q_2(u+1)} + \frac{Q_1(u)Q_1(u+3)}{Q_1(u+1)Q_1(u+2)}\right. \\ &\quad \left. + \frac{Q_1(u+3)Q_2(u)}{Q_1(u+1)Q_2(u+2)}\right).\end{aligned} \tag{8d}$$

We employ the convention such that the eigenvalue $\check{R}_{W_1^{(1)},W_1^{(1)}}(u)$ on the highest component $V_{2\Lambda_1}$ is $[u+3][u+1]$ and let the overall normalization of $\Lambda_1^{(a)}(u)$ as specified by (8). (The common factor $\phi_s^{(2)}(u+\frac{3}{2})$ in (8*c*) has been attached so as to simplify the forthcoming formula (12).) The $\Lambda_1^{(a)}(u)$ consists of $\dim W_1^{(a)} = 4, 5 (a = 1, 2)$ terms and its pole-free conditions are given by (7) in accordance with the analytic Bethe ansatz. The formulae (8) coincide with those in [16, 17] for some special cases. In particular, ratio of Q_a's in $\Lambda_1^{(1)}(u)$ are just those appearing in [16] for the $C_2^{(1)}$ vertex model with $W_s^{(p)} = W_1^{(1)}$ (upon some convention adjustment). Namely, the Q_a-part is determined only from the auxiliary space choice, while the quantum space dependence enters $\phi_s^{(p)}$-part. This is also the case in the formula (3.17) of [8] for the $sl(n)$ case. Similarly, Q_a-part in $\Lambda_1^{(2)}(u)$ are those appearing in the $B_2^{(1)}$ case of [16] due to the equivalence $C_2 \simeq B_2$.

To proceed to $\Lambda_m^{(a)}(u)$ with higher m, we introduce a few more notations.

$$G_a(u) = \begin{cases} \phi_s^{(1)}(u)G(u) & \text{for } a=1 \\ G(u) & \text{for } a=2 \end{cases} \qquad H_a(u) = \begin{cases} H(u) & \text{for } a=1 \\ \phi_2^{(2)}(u)H(u) & \text{for } a=2 \end{cases}$$
$$G(u) = \frac{Q_2(u+\frac{1}{2})Q_2(u-\frac{1}{2})}{Q_1(u+\frac{1}{2})Q_1(u-\frac{1}{2})} \qquad H(u) = \frac{Q_1(u)}{Q_2(u+1)Q_2(u-1)}. \tag{9}$$

We consider the T-system (5) for $\Lambda_m^{(a)}(u)$ with the initial condition for $m=1$ as (8) and

$$\Lambda_0^{(1)}(u) = \phi_s^{(1)}(u+5/2)\phi_s^{(1)}(u+1/2) \qquad \Lambda_0^{(2)}(u) = \phi_s^{(1)}(u+3/2) \qquad \text{for } p=1$$
$$\Lambda_0^{(1)}(u) = \Lambda_0^{(2)}(u) = \phi_s^{(2)}(u+1)\phi_s^{(2)}(u+2) \qquad \text{for } p=2. \tag{10}$$

Then our main result is:

Theorem. The functions

$$\Lambda_m^{(1)}(u) = Q_1\left(u-\frac{m}{2}\right)Q_1\left(u+\frac{m}{2}+3\right)\sum_{0\leqslant i\leqslant j\leqslant m}\ \sum_{l=\left[\frac{i+1}{2}\right]}^{\left[\frac{j+1}{2}\right]}\ \sum_{k=\left[\frac{i}{2}\right]}^{\left[\frac{j}{2}\right]} G_p\left(u+\frac{m+5}{2}-i\right)$$
$$\times G_p\left(u+\frac{m+1}{2}-j\right)H_p\left(u+\frac{m}{2}-2l+2\right)H_p\left(u+\frac{m}{2}-2k+1\right)$$

$$\Lambda_m^{(2)}(u) = Q_2(u-m)Q_2(u+m+3)\sum_{j=0}^{2m}\sum_{l=0}^{\left[\frac{j}{2}\right]}\sum_{k=\left[\frac{j+1}{2}\right]}^{m} G_p\left(u+m+\frac{3}{2}-j\right)H_p(u+m-2l+2)$$
$$\times H_p(u+m-2k+1) \tag{11}$$

are the solutions to the T-system (5) with the initial condition (8), (10) and $g_m^{(a)}(u)$ given by

$$g_m^{(a)}(u) = \begin{cases} \phi_s^{(p)}(u+(m/t_p)+3)\phi_s^{(p)}(u-(m/t_p)) & \text{if } a=p \\ 1 & \text{otherwise.} \end{cases} \tag{12}$$

The symbol $[x]$ in (11) denotes the greatest integer not exceeding x and should not be confused with the one in (6). The function (12) satisfies $g_m^{(a)}(u-(1/t_a))g_m^{(a)}(u+(1/t_a)) = g_{m+1}^{(a)}(u)g_{m-1}^{(a)}(u)$ in accordance with (3.18) of [2] (with a slight normalization change in u). The theorem can be proved by comparing the coefficients of $\phi_s^{(a)}$ factors on both sides of the T-system. In particular, the check essentially reduces to the case $p=s=1$. A similar formula to (11) is available for the $sl(n)$ case in [8].

$\Lambda_m^{(a)}(u)$ (11) Yang–Baxterizes the character of $W_m^{(a)}$ viewed as a C_2-module as in (1). Namely, it contains $\dim W_m^{(a)}$ terms and tends to the latter in the 'braid limit' as follows.

$$\lim_{u\to\infty,(|q|>1)} q^{-\psi_a}\Lambda_m^{(a)}(u) = \chi_m^{(a)}(q^{(\omega^{(p)},\alpha_1)}, q^{(\omega^{(p)},\alpha_2)})$$

$$\psi_a = s\left(2Nu + 3N - 2\sum_{j=1}^{N} w_j\right)\min(1, (t_a/t_p))$$

$$\begin{aligned}\chi_m^{(1)}(z_1, z_2) &= \sum_{0\leqslant i\leqslant j\leqslant m}\sum_{l=\left[\frac{i+1}{2}\right]}^{\left[\frac{j+1}{2}\right]}\sum_{k=\left[\frac{i}{2}\right]}^{\left[\frac{j}{2}\right]} z_1^{2m-2i-2j}z_2^{m-2l-2k} \\ &= \mathrm{ch}\, V_{m\omega_1} + \mathrm{ch}\, V_{(m-2)\omega_1} + \ldots + \begin{cases}1 & m \text{ even}\\ \mathrm{ch}\, V_{\omega_1} & m \text{ odd}\end{cases}\end{aligned} \tag{13}$$

$$\chi_m^{(2)}(z_1, z_2) = \sum_{j=0}^{2m}\sum_{l=0}^{\left[\frac{j}{2}\right]}\sum_{k=\left[\frac{j+1}{2}\right]}^{m} z_1^{2m-2j}z_2^{2m-2l-2k} = \mathrm{ch}\, V_{m\omega_2}.$$

Here, $ch V_\omega = \mathrm{ch}\, V_\omega(z_1, z_2)$ is the irreducible C_2 character with highest weight ω counting the $(\xi\alpha_1 + \eta\alpha_2)$-weight vectors as $z_1^{2\xi}z_2^{2\eta}$. The following character identity [18] is a simple corollary of the above theorem.

$$\begin{aligned}\chi_{2m}^{(1)2} &= \chi_{2m+1}^{(1)}\chi_{2m-1}^{(1)} + \chi_m^{(2)2}\\ \chi_{2m+1}^{(1)2} &= \chi_{2m+2}^{(1)}\chi_{2m}^{(1)} + \chi_m^{(2)}\chi_{m+1}^{(2)}\\ \chi_m^{(2)2} &= \chi_{m+1}^{(2)}\chi_{m-1}^{(2)} + \chi_{2m}^{(1)}.\end{aligned} \tag{14}$$

In [2, 10], $\chi_m^{(a)}$ was denoted by $Q_m^{(a)}$ and (14) was called the Q-system. As shown therein, the combinations $y_{2m}^{(1)}(u) = (g_{2m}^{(1)}(u)\Lambda_m^{(2)}(u-\frac{1}{2})\Lambda_m^{(2)}(u+\frac{1}{2})/\Lambda_{2m+1}^{(1)}(u)\Lambda_{2m-1}^{(1)}(u))$ etc from (5) yield a solution to the $C_2^{(1)}Y$-system [19], the TBA equation in high temperature limit:

$$\begin{aligned}y_{2m}^{(1)}\left(u+\frac{1}{2}\right)y_{2m}^{(1)}\left(u-\frac{1}{2}\right) &= \frac{1+y_m^{(2)}(u)}{(1+y_{2m-1}^{(1)}(u)^{-1})(1+y_{2m+1}^{(1)}(u)^{-1})}\\ y_{2m+1}^{(1)}\left(u+\frac{1}{2}\right)y_{2m+1}^{(1)}\left(u-\frac{1}{2}\right) &= \frac{1}{(1+y_{2m+2}^{(1)}(u)^{-1})(1+y_{2m}^{(1)}(u)^{-1})}\\ y_m^{(2)}(u+1)y_m^{(2)}(u-1) &= \frac{(1+y_{2m-1}^{(1)}(u))(1+y_{2m}^{(1)}(u+\frac{1}{2}))(1+y_{2m}^{(1)}(u-\frac{1}{2}))(1+y_{2m+1}^{(1)}(u))}{(1+y_{m-1}^{(2)}(u)^{-1})(1+y_{m+1}^{(2)}(u)^{-1})}.\end{aligned} \tag{15}$$

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References

[1] Baxter R J 1982 *Exactly Solved Models in Statistical Mechanics* (London: Academic)

[2] Kuniba A, Nakanishi T and Suzuki J 1993 Functional relations in solvable lattice models I: Functional relations and representation theory *Preprint* HUTP-93/A022, hep-th.9309137

[3] Drinfel'd V G 1987 *Proceedings of the ICM* Berkeley (Providence, RI: American Mathematical Society)

[4] Jimbo M 1985 *Lett. Math. Phys.* **10** 63

[5] Kulish P P and Sklyanin E K 1982 *Lecture Notes in Physics 151* (Berlin: Springer)

[6] Baxter R J and Pearce P A 1982 *J. Phys. A: Math. Gen.* **15** 897

[7] Bazhanov V V and Reshetikhin N Yu 1989 *Int. J. Mod. Phys.* A **4** 115

[8] Bazhanov V V and Reshetikhin N Yu 1990 *J. Phys. A: Math. Gen.* **23** 1477

[9] Klümper A and Pearce P A 1992 *Physica* **183A** 304

[10] Kuniba A, Nakanishi T and Suzuki J 1993 Functional relations in solvable lattice models II: Applications *Preprint* HUTP-93/A023, hep-th.9310060

[11] Baxter R J 1972 *Ann. Phys.* **70** 193

[12] Reshetikhin N Yu 1983 *Sov. Phys.–JETP* **57** 691

[13] Bazhanov V V 1985 *Phys. Lett.* B **159** 321

[14] Jimbo M 1986 *Commun. Math. Phys.* **102** 537

[15] Kulish P P, Reshetikhin N Yu and Sklyanin E K 1981 *Lett. Math. Phys.* **5** 393

[16] Reshetikhin N Yu 1987 *Lett. Math. Phys.* **14** 235

[17] Reshetikhin N Yu 1985 *Theor. Math. Phys.* **63** 347

[18] Kirillov A N and Reshetikhin N Yu 1990 *J. Sov. Math.* **52** 3156

[19] Kuniba A and Nakanishi T 1992 *Mod. Phys. Lett.* A **7** 3487